Generalized local equilibrium in the cascaded lattice Boltzmann method

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By realizing the insufficient degree of Galilean invariance of the traditional multiple-relaxation-time collision operators, Geier *et al.* [Phys. Rev. E **73**, 066705 (2006)] proposed to relax differently the moments shifted by the macroscopic velocity, leading to the so-called cascaded lattice Boltzmann method (LBM). This paper points out that (a) the cascaded LBM essentially consists in adopting a generalized local equilibrium in the frame at rest; (b) this new equilibrium does not affect the consistency of LBM; and finally (c) if the raw moments are relaxed in the frame at rest as usual and the number of relaxation frequencies is reduced, the proposed derivation leads to the two-relaxation-time collisional operator with proper polynomial equilibrium.

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I. INTRODUCTION

The lattice Boltzmann method (LBM) is considered a viable alternative for solving the hydrodynamic Navier-Stokes equations [1–3]. The Lax equivalence theorem illustrates that (a) consistency and (b) stability are two essential conditions for ensuring the convergence of the numerical solution to the well-posed initial value problem [4]. (a) The consistency of the LBM with regard to the Navier-Stokes equations can be established, for example, by the Chapman–Enskog expansion [5,6] or by the Hilbert expansion with proper scaling [7,8]. Unfortunately, a mathematical tool for analyzing in general (b) the stability of a system of nonlinear partial differential equations does not exist at present. A popular approach consists of linearizing the system of equations around an arbitrary configuration, applying a Fourier transform in order to get rid of the spatial gradients in the case of periodic boundaries, and finally discussing the obtained ordinary differential equations by von Neumann analysis [4,9,10]. However, in general, many heuristic issues are proposed for guiding the design of stable LBM schemes, including how to discretize the velocity space [11–15] and how to truncate the polynomial expansion of the local equilibrium [14].

The collision step of the algorithm has been proven to play an essential role. In particular, the multiple-relaxation-time (MRT) collisional operator, which was first heuristically proposed in order to enhance collisions [16] and then systematically developed [9,17], and its variants, such as the two-relaxation-time (TRT) operator [18], allow one to enhance the stability by properly tuning the numerical bulk viscosity, which is a free parameter in a scheme aiming to recover the incompressible limit of Navier-Stokes equations.

Recently, a new result was added to the previous picture. By realizing the insufficient degree of Galilean invariance of the traditional MRT collision operators, Geier *et al.* [19] proposed to relax differently the *central moments*, i.e., the moments shifted by the macroscopic velocity, in a *moving frame* (instead of the traditional practice of relaxing the raw moments in the frame at rest), leading to the so-called cascaded LBM.

This paper aims to provide a simple mathematical interpretation, pointing out that (a) the cascaded LBM consists essentially in adopting a *generalized local equilibrium* in the frame at rest, which is a function of both conserved and nonconserved hydrodynamic moments. Moreover (b) the asymptotic analysis proves that the method consistently recovers the correct system of macroscopic equations. Finally (c), despite the different formalism, if the raw moments are relaxed in the frame at rest as usual and the number of relaxation frequencies is reduced, the proposed derivation leads to the TRT collisional operator with proper polynomial equilibrium.

This paper is organized as follows. In Sec. II, some preliminaries are introduced. In Sec. III, two derivations are reported, based on relaxing the raw moments in the frame at rest as usual (result c) and on relaxing the central moments in the moving frame, leading to the cascaded LBM and the generalized local equilibrium (result a). In Sec. IV, it is proven that the generalized local equilibrium does not affect the consistency of the LBM (result b). Finally, some conclusions are reported.

II. PRELIMINARIES

A. Continuous velocity space

Let us introduce the local equilibrium distribution function φ_{eq} in the continuous two-dimensional velocity space $(\xi_x,\xi_y)\in\mathbb{R}^2$, namely,

$$\varphi_{eq}=3\bar{\rho}/(2\pi)\exp\left[-3(\xi_i-\bar{u}_i)^2/2\right], \tag{1}$$

where $\bar{\rho}=\langle\langle\varphi\rangle\rangle$, $\bar{\rho}\bar{u}_i=\langle\langle\xi_i\varphi\rangle\rangle$ $(i=x,y)$, φ is the generic distribution function, and

$$\langle\langle\cdot\rangle\rangle=\int_{-\infty}^{+\infty}d\xi_x d\xi_y. \tag{2}$$

It is possible to prove that the continuous local equilibrium given by Eq. (1) minimizes an entropy function $H(\varphi)$ under the constraints of mass and momentum conservation [14].

Let us introduce the generic continuous *raw* equilibrium moment

$$\gamma^{eq}_{xx\cdots xyy\cdots y}(\overbrace{xx\cdots x}^{n\text{ times}},\overbrace{yy\cdots y}^{m\text{ times}})=\langle\langle\xi_x^n\xi_y^m\varphi_{eq}\rangle\rangle, \tag{3}$$

and the corresponding continuous *central* equilibrium moment

$$\hat{\gamma}^{eq}_{xx\cdots x\,yy\cdots y}(\overbrace{xx\cdots x}^{n\text{ times}},\overbrace{yy\cdots y}^{m\text{ times}}) = \langle\langle(\xi_x-\bar{u}_x)^n(\xi_y-\bar{u}_y)^m\varphi_{eq}\rangle\rangle. \tag{4}$$

In particular, taking into account Eq. (1), it is immediate to realize that the first even central moments are $\hat{\gamma}^{eq}=\bar{\rho}$, $\hat{\gamma}^{eq}_{xx}=\hat{\gamma}^{eq}_{yy}=\bar{\rho}/3$, $\hat{\gamma}^{eq}_{xy}=0$, $\hat{\gamma}^{eq}_{xxyy}=\bar{\rho}/9$, while the first odd central moments are $\hat{\gamma}^{eq}_x=\hat{\gamma}^{eq}_y=\hat{\gamma}^{eq}_{xxy}=\hat{\gamma}^{eq}_{yyx}=0$.

B. Discrete velocity space

Concerning the discrete velocity space, let us consider the D2Q9 lattice, where the discrete velocity component v_i has the following values:

$$v_x=[0,-1,-1,-1,0,1,1,1,0]^T,$$

$$v_y=[0,1,0,-1,-1,-1,0,1,1]^T.$$

Before proceeding, let us define the rule of computation for the lists. Let h and g be the lists defined by $h=[h_0,h_1,h_2,\ldots,h_8]^T$ and $g=[g_0,g_1,g_2,\ldots,g_8]^T$. Then hg is the list defined by $[h_0g_0,h_1g_1,h_2g_2,\ldots,h_8g_8]^T$. The sum of all the elements of the list h is denoted by $\langle h\rangle=\Sigma_{i=0}^8 h_i$.

The equivalent moment space is defined by a transformation matrix, which is not unique. For example, let us consider the *nonorthogonal* transformation matrix, $M=[1;\, v_x;\, v_y;\, v_x^2;\, v_y^2;\, v_xv_y;\, (v_x)^2v_y;\, v_x(v_y)^2;\, (v_x)^2(v_y)^2]^T$, which involves proper combinations of the lattice velocity components. The transformation described by the matrix M diagonalizes the collisional operator of the TRT model (see [18], even though this simple property is not clearly stated there). On the other hand, let us define the following *orthogonal* transformation matrix (considered in [19]):

$$K=\begin{bmatrix} 1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 4\\ 1 & -1 & 1 & 2 & 0 & 1 & -1 & 1 & 1\\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & -2 & -2\\ 1 & -1 & -1 & 2 & 0 & -1 & 1 & 1 & 1\\ 1 & 0 & -1 & -1 & -1 & 0 & -2 & 0 & -2\\ 1 & 1 & -1 & 2 & 0 & 1 & 1 & -1 & 1\\ 1 & 1 & 0 & -1 & 1 & 0 & 0 & 2 & -2\\ 1 & 1 & 1 & 2 & 0 & -1 & -1 & -1 & 1\\ 1 & 0 & 1 & -1 & -1 & 0 & 2 & 0 & -2 \end{bmatrix}, \tag{5}$$

where clearly K^TK is diagonal.

The dimensionless density $\bar{\rho}$ and flow velocity $\bar{u}_i$ are defined by $\bar{\rho}=\langle f\rangle$ and $\bar{\rho}\bar{u}_i=\langle v_i f\rangle$, where f is the discrete distribution function. Let us introduce the generic discrete *raw* moment

$$\pi_{xx\cdots x\,yy\cdots y}(\overbrace{xx\cdots x}^{n\text{ times}},\ \overbrace{yy\cdots y}^{m\text{ times}}) = \langle v_x^n v_y^m f\rangle, \tag{6}$$

and the corresponding generic discrete *central* moment

$$\hat{\pi}_{xx\cdots x\,yy\cdots y}(\overbrace{xx\cdots x}^{n\text{ times}},\overbrace{yy\cdots y}^{m\text{ times}}) = \langle(v_x-\bar{u}_x)^n(v_y-\bar{u}_y)^m f\rangle. \tag{7}$$

III. CASCADED LBM

The generic LBM algorithm consists of a *collision process* and a *streaming process*. Following [19], we define the *collision process* as

$$f^p = f + Kg(f,f_{eq},\lambda_e,\lambda_o), \tag{8}$$

where f_{eq} is the discrete local equilibrium, λ_e and λ_o are the relaxation frequencies for the even and odd moments, respectively, and f^p is the postcollision distribution function. All the previous quantities are computed in $(\bar{t},\bar{x}_i,v_i)$, where $\bar{t}$ and $\bar{x}_i$ are the time and space in lattice units, respectively. We define the *streaming step* as $f(\bar{t}+1,\bar{x}_i+v_i,v_i)=f^p(\bar{t},\bar{x}_i,v_i)$.

Because of the collisional invariants, $g_0=g_1=g_2=0$. Concerning the remaining terms g_α ($\alpha=3-8$), following [19], let us consider first the particular case $\lambda_e=\lambda_o=1$, which implies that the postcollision distribution function is in equilibrium, namely,

$$f^p_{eq} = f + Kg^*, \tag{9}$$

where $g^*=g(f,f_{eq},1,1)$. Let us multiply Eq. (9) by $(v_x-\bar{u}_x)^n(v_y-\bar{u}_y)^m$, let us take the sum $\langle\cdot\rangle$ of the resulting list and, finally, let us assume that the equilibrium moments of the postcollision discrete function coincide with the continuous counterparts, namely,

$$\langle(v_x-\bar{u}_x)^n(v_y-\bar{u}_y)^m Kg^*_\alpha\rangle = \hat{\gamma}^{eq}_{xx\cdots xyy\cdots y} - \hat{\pi}_{xx\cdots xyy\cdots y}, \tag{10}$$

where $\alpha=3-8$. In particular, considering the first moments (discussed in Sec. II) and realizing that the left-hand side of Eq. (10) is linear with regard to g^*_α ($\alpha=3-8$) yields

$$S\begin{bmatrix} g_3^*\\ g_4^*\\ g_5^*\\ g_6^*\\ g_7^*\\ g_8^* \end{bmatrix} = \begin{bmatrix} \hat{\gamma}^{eq}_{xx}-\hat{\pi}_{xx}\\ \hat{\gamma}^{eq}_{yy}-\hat{\pi}_{yy}\\ \hat{\gamma}^{eq}_{xy}-\hat{\pi}_{xy}\\ \hat{\gamma}^{eq}_{xxy}-\hat{\pi}_{xxy}\\ \hat{\gamma}^{eq}_{xyy}-\hat{\pi}_{xyy}\\ \hat{\gamma}^{eq}_{xxyy}-\hat{\pi}_{xxyy} \end{bmatrix}, \tag{11}$$

where S is the shift matrix for passing from the *frame at rest* to the *moving frame*, namely,

$$S=\begin{bmatrix} 6 & 2 & 0 & 0 & 0 & 0\\ 6 & -2 & 0 & 0 & 0 & 0\\ 0 & 0 & -4 & 0 & 0 & 0\\ -6\bar{u}_y & -2\bar{u}_y & 8\bar{u}_x & -4 & 0 & 0\\ -6\bar{u}_x & 2\bar{u}_x & 8\bar{u}_y & 0 & -4 & 0\\ 8+6(\bar{u}_x^2+\bar{u}_y^2) & 2(\bar{u}_y^2-\bar{u}_x^2) & -16\bar{u}_x\bar{u}_y & 8\bar{u}_y & 8\bar{u}_x & 4 \end{bmatrix}, \tag{12}$$

while the vector on the right-hand side of Eq. (11) is

$$\begin{bmatrix} \hat{\pi}_{xx} \\ \hat{\pi}_{yy} \\ \hat{\pi}_{xy} \\ \hat{\pi}_{xxy} \\ \hat{\pi}_{xyy} \\ \hat{\pi}_{xxyy} \end{bmatrix} = \begin{bmatrix} \pi_{xx} - \bar{\rho}\bar{u}_x^2 \\ \pi_{yy} - \bar{\rho}\bar{u}_y^2 \\ \pi_{xy} - \bar{\rho}\bar{u}_x\bar{u}_y \\ \pi_{xxy} - \pi_{xx}\bar{u}_y - 2\bar{u}_x\pi_{xy} + 2\bar{\rho}\bar{u}_x^2\bar{u}_y \\ \pi_{xyy} - 2\pi_{xy}\bar{u}_y - \bar{u}_x\pi_{yy} + 2\bar{\rho}\bar{u}_x\bar{u}_y^2 \\ \pi_{xxyy} - 2\pi_{xxy}\bar{u}_y - 2\bar{u}_x\pi_{xyy} + \pi_{xx}\bar{u}_y^2 + \bar{u}_x^2\pi_{yy} + 4\bar{u}_x\bar{u}_y\pi_{xy} - 3\bar{\rho}\bar{u}_x^2\bar{u}_y^2 \end{bmatrix}. \tag{13}$$

Solving the system of equations given by Eq. (11) yields

$$\begin{bmatrix} g_3^* \\ g_4^* \\ g_5^* \\ g_6^* \\ g_7^* \\ g_8^* \end{bmatrix} = \begin{bmatrix} -(\pi_{xx}+\pi_{yy})/12 + \bar{\rho}/18 + \bar{\rho}\bar{u}_x^2/12 + \bar{\rho}\bar{u}_y^2/12 \\ -(\pi_{xx}-\pi_{yy})/4 + \bar{\rho}\bar{u}_x^2/4 - \bar{\rho}\bar{u}_y^2/4 \\ \pi_{xy}/4 - \bar{\rho}\bar{u}_x\bar{u}_y/4 \\ \pi_{xxy}/4 - \bar{\rho}\bar{u}_y/12 - \bar{\rho}\bar{u}_x^2\bar{u}_y/4 \\ \pi_{xyy}/4 - \bar{\rho}\bar{u}_x/12 - \bar{\rho}\bar{u}_x\bar{u}_y^2/4 \\ (\pi_{xx}+\pi_{yy})/6 - \pi_{xxyy}/4 - \bar{\rho}/12 - \bar{\rho}\bar{u}_x^2/12 - \bar{\rho}\bar{u}_y^2/12 + \bar{\rho}\bar{u}_x^2\bar{u}_y^2/4 \end{bmatrix}. \tag{14}$$

A. Recovering the traditional TRT scheme

Before proceeding with the derivation reported in [19], let us consider first the particular choice $g_3=\lambda_e g_3^*$, $g_4=\lambda_e g_4^*$, $g_5=\lambda_e g_5^*$, $g_6=\lambda_o g_6^*$, $g_7=\lambda_o g_7^*$, and $g_8=\lambda_e g_8^*$. In this case, Eq. (8) can be rewritten in a simpler way,

$$f^p = f + Kg = f + M^{-1}(MKg) = f + A(f_{\mathrm{eq}} - f), \tag{15}$$

where $A=M^{-1}\Lambda M$,

$$\Lambda = \mathrm{diag}([0,0,0,\lambda_e,\lambda_e,\lambda_e,\lambda_o,\lambda_o,\lambda_e]),$$

and

$$Mf_{\mathrm{eq}} = \begin{bmatrix} \pi^{\mathrm{eq}} \\ \pi_x^{\mathrm{eq}} \\ \pi_y^{\mathrm{eq}} \\ \pi_{xx}^{\mathrm{eq}} \\ \pi_{yy}^{\mathrm{eq}} \\ \pi_{xy}^{\mathrm{eq}} \\ \pi_{xxy}^{\mathrm{eq}} \\ \pi_{xyy}^{\mathrm{eq}} \\ \pi_{xxyy}^{\mathrm{eq}} \end{bmatrix} = \begin{bmatrix} \bar{\rho} \\ \bar{\rho}\bar{u}_x \\ \bar{\rho}\bar{u}_y \\ \bar{\rho}/3 + \bar{\rho}\bar{u}_x^2 \\ \bar{\rho}/3 + \bar{\rho}\bar{u}_y^2 \\ \bar{\rho}\bar{u}_x\bar{u}_y \\ \bar{\rho}\bar{u}_y/3 + \bar{\rho}\bar{u}_x^2\bar{u}_y \\ \bar{\rho}\bar{u}_x/3 + \bar{\rho}\bar{u}_x\bar{u}_y^2 \\ \bar{\rho}/9 + \bar{\rho}/3(\bar{u}_x^2+\bar{u}_y^2) + \bar{\rho}\bar{u}_x^2\bar{u}_y^2 \end{bmatrix} = \begin{bmatrix} \gamma^{\mathrm{eq}} \\ \gamma_x^{\mathrm{eq}} \\ \gamma_y^{\mathrm{eq}} \\ \gamma_{xx}^{\mathrm{eq}} \\ \gamma_{yy}^{\mathrm{eq}} \\ \gamma_{xy}^{\mathrm{eq}} \\ \gamma_{xxy}^{\mathrm{eq}} \\ \gamma_{xyy}^{\mathrm{eq}} \\ \gamma_{xxyy}^{\mathrm{eq}} \end{bmatrix}. \tag{16}$$

The previous expressions are perfectly equivalent to the TRT scheme with $c_s^2=\frac{1}{3}$ [18], which has bulk viscosity equal to kinematic viscosity (as explained in Sec. 2.1 of [20]). The previous polynomial equilibrium has the same moments of the continuous Maxwellian, given by Eq. (3). It is possible to prove that A is exactly the collisional matrix of the TRT scheme and f_{eq} is the Taylor expansion of the continuous equilibrium given by Eq. (1) for the D2Q9 lattice. If the terms higher than second order with regard to macroscopic velocity were neglected, then the previous equilibrium would reduce to the standard expression, which is sufficient for consistency [8].

Hence, if the raw moments are relaxed in the frame at rest as usual and only two relaxation frequencies are considered, the proposed derivation leads to the TRT collisional operator with proper polynomial equilibrium (result c).

B. Recovering the cascaded LBM scheme

The previous choice of the relaxation process recovering the TRT scheme can be interpreted in terms of the following definitions of g_α (α=3–8):

$$\begin{bmatrix} g_3/\lambda_e \\ g_4/\lambda_e \\ g_5/\lambda_e \\ g_6/\lambda_o \\ g_7/\lambda_o \\ g_8/\lambda_e \end{bmatrix} = S^{-1} \begin{bmatrix} \hat{\gamma}_{xx}^{eq} - \hat{\pi}_{xx} \\ \hat{\gamma}_{yy}^{eq} - \hat{\pi}_{yy} \\ \hat{\gamma}_{xy}^{eq} - \hat{\pi}_{xy} \\ \hat{\gamma}_{xxy}^{eq} - \hat{\pi}_{xxy} \\ \hat{\gamma}_{xyy}^{eq} - \hat{\pi}_{xyy} \\ \hat{\gamma}_{xxyy}^{eq} - \hat{\pi}_{xxyy} \end{bmatrix}, \tag{17}$$

where S^{-1} is the shift matrix for passing from the *moving frame* to the *frame at rest*. Clearly in the previous expression, the relaxation is done in the frame at rest. In order to relax the central moments, i.e., the moments shifted by the macroscopic velocity, in the moving frame, it is enough to apply the relaxation frequencies *before* multiplying by S^{-1}.

Actually, in Ref. [19], the relaxation is done neither in the frame at rest nor in the moving frame, but the cascaded relaxation is defined instead. First of all, the particular choice $g_3'=\lambda_e^\xi g_3^*$, $g_4'=\lambda_e^\nu g_4^*$, $g_5'=\lambda_e^\nu g_5^*$ is assumed (which is equivalent to relaxing the stress tensor components in the frame at rest), where λ_e^ν is the relaxation frequency controlling the kinematic viscosity and λ_e^ξ is that controlling the bulk viscosity. By means of the fourth and fifth rows of matrix S defined by Eq. (12), the quantities g_6' and g_7' are computed, namely,

$$-6\bar{u}_y g_3' - 2\bar{u}_y g_4' + 8\bar{u}_x g_5' - 4g_6' = \lambda_o(\hat{\gamma}_{xxy}^{eq} - \hat{\pi}_{xxy}), \tag{18}$$

$$-6\bar{u}_x g_3' + 2\bar{u}_x g_4' + 8\bar{u}_y g_5' - 4g_7' = \lambda_o(\hat{\gamma}_{xyy}^{eq} - \hat{\pi}_{xyy}), \tag{19}$$

and, by means of the last row of matrix S, the quantity g_8' is computed, namely,

$$[8+6(\bar{u}_x^2+\bar{u}_y^2)]g_3' + 2(\bar{u}_y^2-\bar{u}_x^2)g_4' - 16\bar{u}_x\bar{u}_y g_5' + 8\bar{u}_y g_6' + 8\bar{u}_x g_7' + 4g_8' = \lambda_e(\hat{\gamma}_{xxyy}^{eq} - \hat{\pi}_{xxyy}). \tag{20}$$

The previous choice is equivalent to relaxing the higher-order moments in the moving frame.

Also in this case, it is possible to search for a simplified evolution equation, namely,

$$f'^p = f + Kg' = f + M^{-1}(MKg') = f + A'(f_{eq}' - f), \tag{21}$$

where $A'=M^{-1}\Lambda' M$ and Λ' is the block-diagonal matrix defined as

$$\Lambda' = \text{diag}\left([0,0,0], \begin{bmatrix} \lambda_e^+ & \lambda_e^- \\ \lambda_e^- & \lambda_e^+ \end{bmatrix}, [\lambda_e^\nu, \lambda_o, \lambda_o, \lambda_e]\right), \tag{22}$$

where $\lambda_e^+=(\lambda_e^\xi+\lambda_e^\nu)/2$ and $\lambda_e^-=(\lambda_e^\xi-\lambda_e^\nu)/2$, while the moments of f_{eq}' are identical to those of f_{eq} reported in Eq. (16), with the exception of

$$\pi_{xxy}'^{eq} = \pi_{xxy}^{eq} + (1-\omega_\xi)/2\bar{u}_y[(\pi_{xx}-\pi_{xx}^{eq}) + (\pi_{yy}-\pi_{yy}^{eq})] + (1-\omega_\nu)/2\bar{u}_y[(\pi_{xx}-\pi_{xx}^{eq}) - (\pi_{yy}-\pi_{yy}^{eq})] + 2(1-\omega_\nu)\bar{u}_x(\pi_{xy}-\pi_{xy}^{eq}), \tag{23}$$

$$\pi_{xyy}'^{eq} = \pi_{xyy}^{eq} + (1-\omega_\xi)/2\bar{u}_x[(\pi_{yy}-\pi_{yy}^{eq}) + (\pi_{xx}-\pi_{xx}^{eq})] + (1-\omega_\nu)/2\bar{u}_x[(\pi_{yy}-\pi_{yy}^{eq}) - (\pi_{xx}-\pi_{xx}^{eq})] + 2(1-\omega_\nu)(\pi_{xy}-\pi_{xy}^{eq})\bar{u}_y, \tag{24}$$

$$\begin{aligned}\pi_{xxyy}'^{eq} = {} & \pi_{xxyy}^{eq} + 2(1-\theta)[\bar{u}_x(\pi_{xyy}-\pi_{xyy}^{eq}) + (\pi_{xxy}-\pi_{xxy}^{eq})\bar{u}_y] \\ & - 2(1-\theta)[\bar{u}_x^2(\pi_{yy}-\pi_{yy}^{eq}) + (\pi_{xx}-\pi_{xx}^{eq})\bar{u}_y^2 \\ & + 4\bar{u}_x\bar{u}_y(\pi_{xy}-\pi_{xy}^{eq})] + (1-\theta_\xi)/2\{(\bar{u}_x^2+\bar{u}_y^2) \\ & \times[(\pi_{yy}-\pi_{yy}^{eq}) + (\pi_{xx}-\pi_{xx}^{eq})]\} + (1-\theta_\nu)/2\{(\bar{u}_x^2-\bar{u}_y^2) \\ & \times[(\pi_{yy}-\pi_{yy}^{eq}) - (\pi_{xx}-\pi_{xx}^{eq})]\} \\ & + 4(1-\theta_\nu)\bar{u}_x\bar{u}_y(\pi_{xy}-\pi_{xy}^{eq}),\end{aligned} \tag{25}$$

where $\omega_\nu=\lambda_e^\nu/\lambda_o$, $\omega_\xi=\lambda_e^\xi/\lambda_o$, $\theta=\lambda_o/\lambda_e$, $\theta_\nu=\lambda_e^\nu/\lambda_e$, and $\theta_\xi=\lambda_e^\xi/\lambda_e$. Clearly in the case of single relaxation time, $\omega_\nu=\omega_\xi=\theta=\theta_\nu=\theta_\xi=1$ and $f_{eq}'=f_{eq}$, proving that, for the Bhatnagar-Gross-Krook (BGK) scheme [1], the cascaded relaxation coincides with the relaxation of the raw moments in the frame at rest. However, in general, relaxing differently the central moments in the moving frame is equivalent to considering a *generalized local equilibrium*, depending on both conserved (as it happens in kinetic theory) and nonconserved moments, such as π_{ij} and π_{ijk}, in the frame at rest (result a). Clearly the opposite holds as well, because relaxing differently the moments in the frame at rest (as usual) leads to a generalization of the equilibrium in the moving frame. Hence the previous result seems to suggest that, among all the possible relaxations that can be recast in the form given by Eqs. (21)–(25), only the BGK relaxation actually avoids any equilibrium generalization in any frame.

IV. GRAD MOMENT EXPANSION

In order to check that the numerical scheme is actually consistent with regard to the desired incompressible Navier-Stokes equations, let us apply the procedure proposed in Ref. [21] based on the Grad moment expansion.

Let us introduce first the diffusion scaling [7,8]. Introducing the small parameter ϵ as $\epsilon=l_c/L$, which corresponds to the Knudsen number, where l_c is the mean free path and L is a macroscopic characteristic length, we have $x_i=\epsilon\bar{x}_i$. Furthermore, assuming $U/c=\epsilon$, which corresponds to the Mach number, where U is the macroscopic characteristic speed and c is proportional to the sound speed, we have $t=\epsilon^2\bar{t}$. Consequently, plugging the collisional operator given by Eq. (21) in a kinetic evolution equation for f yields

$$\epsilon^2\frac{\partial f}{\partial t} + \epsilon v_i\frac{\partial f}{\partial x_i} = A'(f_{eq}'-f). \tag{26}$$

Taking into account that $\bar{u}_i=\epsilon u_i$ because of the considered low Mach number limit, let us compute the first moments of Eq. (26), namely,

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial(\bar{\rho}u_i)}{\partial x_i} = 0, \tag{27}$$

$$\epsilon^3 \frac{\partial(\bar{\rho} u_i)}{\partial t} + \epsilon \frac{\partial \pi_{ij}}{\partial x_j} = 0, \tag{28}$$

where the stress tensor components satisfy

$$\epsilon^2 \frac{\partial \pi_{xx}}{\partial t} + \epsilon \frac{\partial \pi_{xxk}}{\partial x_k} = \lambda_\sigma^+ (\pi_{xx}^{\mathrm{eq}} - \pi_{xx}) + \lambda_\sigma^- (\pi_{yy}^{\mathrm{eq}} - \pi_{yy}), \tag{29}$$

$$\epsilon^2 \frac{\partial \pi_{yy}}{\partial t} + \epsilon \frac{\partial \pi_{yyk}}{\partial x_k} = \lambda_\sigma^- (\pi_{xx}^{\mathrm{eq}} - \pi_{xx}) + \lambda_\sigma^+ (\pi_{yy}^{\mathrm{eq}} - \pi_{yy}), \tag{30}$$

$$\epsilon^2 \frac{\partial \pi_{xy}}{\partial t} + \epsilon \frac{\partial \pi_{xyk}}{\partial x_k} = \lambda_\sigma^\nu (\pi_{xy}^{\mathrm{eq}} - \pi_{xy}). \tag{31}$$

Since $O(\pi_{ijk}) = O(\pi_{ijk}^{\mathrm{eq}}) = \epsilon$ [21], the previous equations prove that $O(\pi_{ij} - \pi_{ij}^{\mathrm{eq}}) = O(\epsilon^2)$. Introducing this result into Eqs. (23) and (24) yields

$$\pi'^{\mathrm{eq}}_{xxy} - \pi^{\mathrm{eq}}_{xxy} = O(\epsilon^3), \quad \pi'^{\mathrm{eq}}_{xyy} - \pi^{\mathrm{eq}}_{xyy} = O(\epsilon^3).$$

Searching for approximated expressions of the stress tensor components, it is possible to assume that $\pi_{xxy} \sim \pi'^{\mathrm{eq}}_{xxy} \sim \pi^{\mathrm{eq}}_{xxy}$ and $\pi_{xyy} \sim \pi'^{\mathrm{eq}}_{xyy} \sim \pi^{\mathrm{eq}}_{xyy}$ without affecting the second-order accuracy of the method. The generalized local equilibrium differs from the Taylor-expansion-based equilibrium given by Eq. (16) for higher-order terms, which do not modify the recovered macroscopic equations up to the incompressible Navier-Stokes level (result b).

V. CONCLUSIONS

The cascaded LBM [19] represents an alternative approach to enhance the stability of traditional MRT-LBM schemes. The present work shows that the cascaded LBM uses a generalized local equilibrium in the frame at rest, which depends on both conserved and nonconserved moments. This new equilibrium does not affect the consistency of the LBM. These results may clarify the essence of the cascaded LBM and they may help in developing new schemes in a systematic way.

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